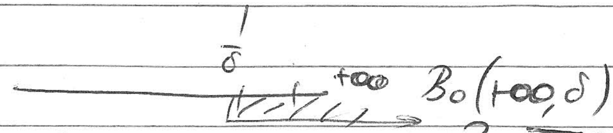
26/10/15

$f : A \to \mathbb{R}$

$A \subseteq \mathbb{R}$



<u>$y \in A''$</u>: $\mathbb{R}^* \ni \limsup_{x \to y} f(x) = \inf_{\delta > 0} \sup \{ f(x) : x \in B_0(y, \delta) \cap A \} = \overline{\lim_y} f$

$$\liminf_{x \to y} f(x) = \sup \inf \{ f(x) : x \in B_0(y, \delta) \cap A \} = \underline{\lim_y} f$$

<u>Ιδιότητες</u>: 1) $\liminf_y (-f) = -\limsup_y f$

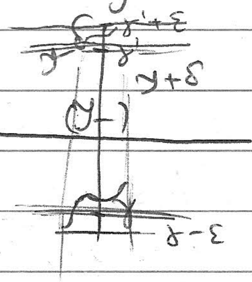
$\limsup_y (-f) = -\liminf_y f$

2) $\liminf_y f \leq \limsup_y f$

3) $\liminf_y f = \limsup_y f \iff \exists \ell \in \mathbb{R}^* : \ell = \lim_y f$

4) $\liminf_y f \geq \gamma \in \mathbb{R} \iff (\forall \varepsilon>0)(\exists \delta>0) : \forall x \in B_0(y,\delta)\cap A$ να ισχύει

$f(x) \geq \gamma - \varepsilon$



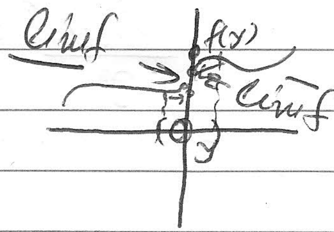
$\limsup_y f \leq \gamma' \in \mathbb{R} \iff (\forall\varepsilon>0)(\exists\delta'>0)\ \forall x\in B_0(y,\delta')\cap A$

$f(x)\leq \gamma+\varepsilon.$

$\longleftrightarrow$

$\exists (x_n)\in A,\ x_n\neq y,\ x_n\to y : f(x_n)\to \limsup_y f$

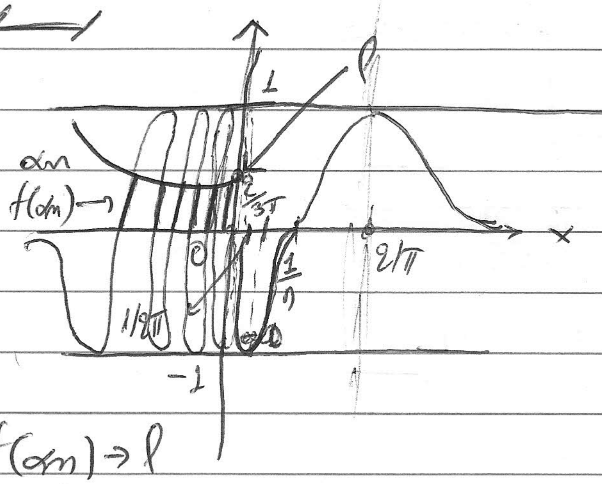
$\exists (x_n')\in A,\ x_n'\neq y,\ x_n'\to y : f(x_n')\to \liminf_y f$



$\longleftrightarrow$

$f(x)=\sin\frac{1}{x},\ x\neq 0$

$y=0$



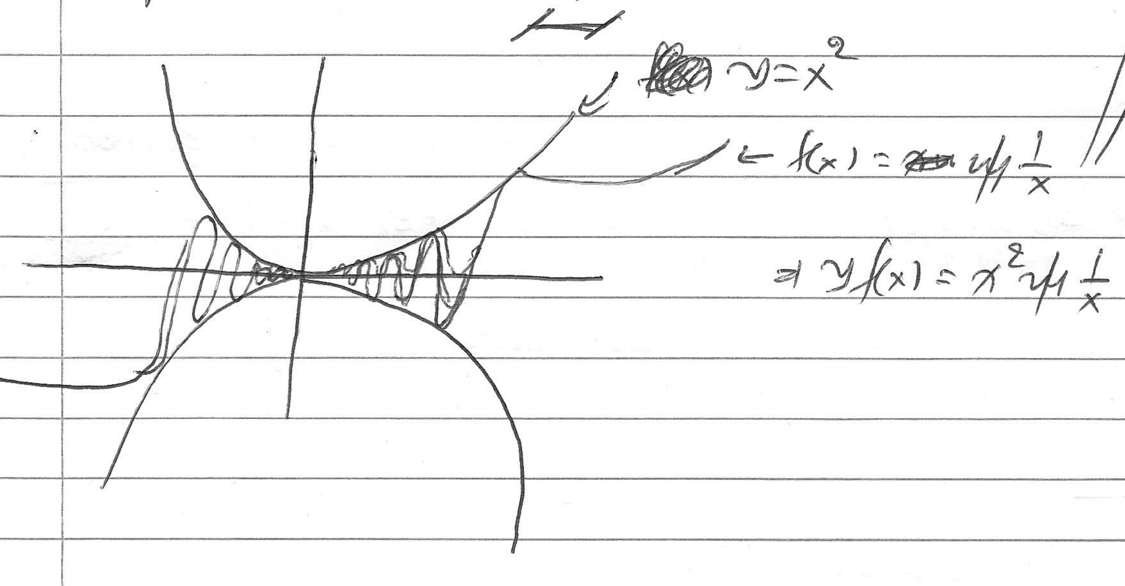
$x_n\neq 0,\ x_n\to 0$

$f(x_n)\to \ell$

$\forall \ell\in[-1,1]\ \exists x_n\neq 0$

με $x_n\to 0$ και $f(x_n)\to \ell$

$\longleftrightarrow$



$$\overline{\lim_{y}}\,(f(x)+g(x)) \le \overline{\lim_{y}}\, f(x) + \overline{\lim_{y}}\, g(x)$$

$$\underline{\lim_{y}}\,(f+g) \ge \underline{\lim_{y}}\, f + \underline{\lim_{y}}\, g$$

$$f,g>0 \Rightarrow \overline{\lim_{y}}\,(fg) \le \overline{\lim}\, f \cdot \overline{\lim}\, g$$

$$\Rightarrow \underline{\lim_{y}}\,(f\cdot g) \le \underline{\lim_{y}}\, f \cdot \underline{\lim_{y}}\, g$$

$A \subseteq \mathbb{R}^n, \quad F(A,\mathbb{R}) = \{f: A \to \mathbb{R}\}$

↓ διανυσματικός χώρος:

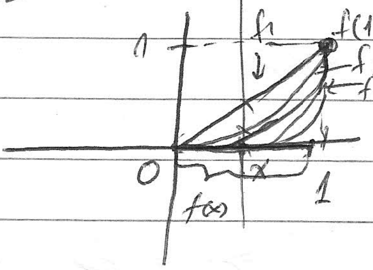
$(f+g)(x) = f(x)+g(x), \; x\in A$

$(\alpha\cdot f)(x) = \alpha \cdot f(x), \; x \in A.$

$f_n \in F(A,\mathbb{R}), \quad f_n(x) \in \mathbb{R}, \quad \lim_{n\to\infty} f_n(x) = f(x) \in \mathbb{R}$

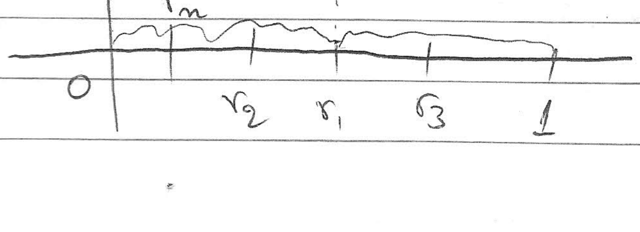
$f_n \xrightarrow{\text{σημ.}} f$ (κατά σημείο).

π.χ. $f_n(x) = x^n, \; x \in [0,1]$



$$\left.\begin{array}{l} \text{Για } x\in[0,1) \Rightarrow f_n(x) \to 0 \\ x=1 \Rightarrow f_n(1) = 1 \to 1 \end{array}\right\} =: f(x)$$

$[0,1] \cap \mathbb{Q} = \{r_1, r_2, \dots, r_n, \dots\}$



$$f_1(x) = \begin{cases} 1, & x = r_1 \\ 0, & \text{αλλού} \end{cases}$$

$$f_2(x) = \begin{cases} 1, & x \in \{r_1, r_2\} \\ 0, & \text{αλλού} \end{cases}$$

$\vdots$

$$f_n(x) = \begin{cases} 1, & x \in \{r_1, \dots, r_n\} \\ 0, & \text{αλλού.} \end{cases}$$

Έστω $\bar{x} \in [0,1] \Rightarrow$ $\bar{x} \in \mathbb{Q} \Rightarrow \bar{x} = r_k$ // $\Rightarrow f_n(\bar{x}) \longrightarrow 1$
$\bar{x} \notin \mathbb{Q}$

αν $n < k \Rightarrow f_n(\bar{x}) = 0$
$n \geq k \Rightarrow f_n(\bar{x}) = 1$

Αν $x \notin \mathbb{Q} \Rightarrow f_n(x) = 0 \Rightarrow f_n(x) \to 0$, $\forall n$.

δηλ $f_n \xrightarrow{\text{σημ.}} f$: $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$

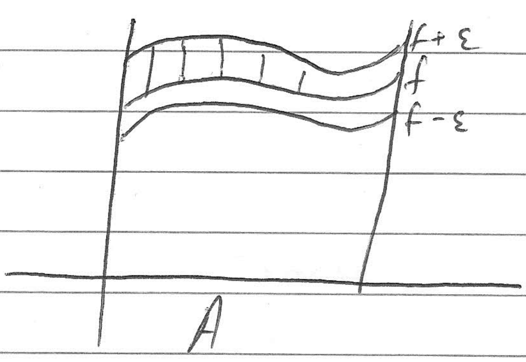
Η f_n συγκλίνει ομοιόμορφα στην f αν $\sup\limits_{x \in A} |f_n(x) - f(x)| \to 0$.

Μηδενική ακολ. : $(\forall \varepsilon > 0)(\exists n_0) : n \geq n_0 \Rightarrow \sup\limits_{x \in A} |f_n(x) - f(x)| < \varepsilon$

δηλ $\underbrace{(\forall \varepsilon > 0)(\exists n_0)\ n \geq n_0 \Rightarrow (\forall x \in A) : |f_n(x) - f(x)| < \varepsilon}_{n_0 = n_0(\varepsilon)}$ ← ομοιόμορφη σύγκλ.

$\lim f_n(x) = f(x) \in \mathbb{R} \iff (\forall \bar{x} \in A)(\forall \varepsilon > 0)(\exists n_0) : n \geq n_0 \Rightarrow$
$\Rightarrow \underbrace{|f_n(x) - f(x)| < \varepsilon}_{n_0 = n_0(\varepsilon, \bar{x})}$. ← κατά σημείο

άρα $\forall x \in \mathbb{R}$: $f(x) - \varepsilon \leq f_n(x) < f(x) + \varepsilon$



δηλ $f_n \to f \Rightarrow f_n \xrightarrow{\text{ομ.}} f$

$N_\varepsilon(x) := \inf\{ n_0 : n \geq n_0 : |f_n(x) - f(x)| < \varepsilon \}$

$\sup\limits_{x \in A} N_\varepsilon(x) = N \Rightarrow N_\varepsilon(x) \leq N$, $\forall x \in A$
$\downarrow$ $\qquad |f_n(x) - f(x)| < \varepsilon$
$N < +\infty$

$y \in A''$

$f_n \to f : A \to \mathbb{R}$, $\exists \ell \in \mathbb{R}^*$, $\lim\limits_{x \to y} f(x) = \ell$

$\forall x_n \in A : x_n \to y$, $f_n(x_n) \to \ell$ (ομοιόμορφη σύγκλιση)
$\exists x_n \in A : x_n \to y$, $f_n(x_n) \not\to \ell$ (όχι ομ. σύγκλιση).

$f_n(x) = x^n$ ημ $\frac{1}{1-x}$, $x \in (0,1)$

κατά σημείο όριο $= 0$ δηλ $f_n \xrightarrow{\text{σημ.}} 0$ ($x<0 \Rightarrow x^n \to 0$ και ημ φραγμ.)

$x_n = 1 - \frac{1}{2n\pi} \to 1 \ (n \to \infty)$, $x_n' = 1 - \frac{1}{2n\pi + \frac{\pi}{2}}$

$f_n(x_n) = \left(1 - \frac{1}{2n\pi}\right)^n$ ημ$(2n\pi) = 0 \to 0$ δηλ $f_n(x_n) \to 0 \ \forall x_n$

$$f_n(x_n') = \left(1 - \frac{1}{2n\pi + \frac{\pi}{2}}\right)^n \cancel{\text{ημ}}\left(\frac{\pi}{2}\right)^{1} = \left(\left(1 - \frac{1}{2n\pi + \frac{\pi}{2}}\right)^{n \cdot 2\pi}\right)^{\frac{1}{2\pi}} \to$$

$$\to e^{-\frac{1}{2\pi}}$$

άρα αφού $\left.\begin{array}{l} f_n(x_n) \to \ell_1 \\ f_n(x_n') \to \ell_2 \end{array}\right| \Rightarrow \ell_1 \neq \ell_2 \Rightarrow$ <u>όχι</u> ομ. σύγκλιση.